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On the Nilpotent Multiplier of a Free Product [*]

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Abstract

In this paper, using a result of J. Burns and G. Ellis (Math. Z. 226(1997) 405-28.), we prove that the c-nilpotent multiplier (the Baer-invariant with respect to the variety of nilpotent groups of class at most c, $\mathcal{N}_c$.) *does commute* with the free product of cyclic groups of mutually coprime order.

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1. Introduction and Motivation

I. Schur [12], in 1904, using projective representation theory of groups, introduced the notion of a multiplier of a finite group. It was known later that the Schur multiplier had a relation with homology and cohomology of groups. In fact, if G is a finite group, then

$$M(G) \cong H^2(G, \mathbf{C}^*) \quad \mathit{and} \quad M(G) \cong H_2(G, \mathbf{Z}) \ ,$$

where $M(G)$ is the Schur multiplier of G, $H^2(G, \mathbf{C}^*)$ is the second cohomology of G with coefficient in $\mathbf{C}^*$ and $H_2(G, \mathbf{Z})$ is the second internal homology of G [see 7]. In 1942, H. Hopf [6] proved that

$$M(G) \cong H^2(G, \mathbf{C}^*) \cong \frac{R \cap F'}{[R, F]} \ ,$$

where G is presented as a quotient $G = F/R$ of a free group F by a normal subgroup R in F. He also proved that the above formula is independent of the presentation of G.

R. Baer [1], in 1945, using the variety of groups, generalized the notion of the Schur multiplier as follows.

Let $\mathcal{V}$ be a variety of groups defined by the set of laws V and let G be a group with a free presentation $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$. Then the *Baer-invariant* of G with respect to the variety $\mathcal{V}$ is defined to be

$$\mathcal{V}M(G) := \frac{R \cap V(F)}{[RV^*F]} \ ,$$

where $V(F)$ is the verbal subgroup of F with respect to $\mathcal{V}$ and

$$[RV^*F] =< v(f_1, \ldots, f_{i-1}, f_i r, f_{i+1}, \ldots, f_n) v(f_1, \ldots, f_i, \ldots f_n)^{-1} \mid r \in R,$$

$$1 \le i \le n, v \in V, f_i \in F, n \in \mathbf{N} > .$$

It is known that the Baer-invariant of a group G is always abelian and independent of the choice of the presentation of G. (See C. R. Leedham-Green and S. McKay [8], from which our notation has been taken, and H.

Neumann [10] for the notion of variety of groups.) Note that if $\mathcal{V}$ is the variety of abelian groups, $\mathcal{A}$, then the Baer-invariant of G will be

$$\mathcal{A}M(G) = \frac{R \cap F'}{[R, F]} ,$$

which is the Schur multiplier of G, $M(G)$. Also if $\mathcal{V} = \mathcal{N}_c$ is the variety of nilpotent groups of class at most $c \geq 1$, then the Baer-invariant of the group G with respect to $\mathcal{N}_c$ will be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ _cF]} ,$$

where $\gamma_{c+1}(F)$ is the $(c+1)$-st term of the lower central series of F and $[R,_1 F] = [R, F]$, $[R,_c F] = [[R,_{c-1} F], F]$. According to J. Burns and G. Ellis' paper [2] we shall call $\mathcal{N}_c M(G)$ the c-nilpotent multiplier of G and denote it by $M^{(c)}(G)$. It is easy to see that 1-nilpotent multiplier is actually the Schur multiplier.

Theorem 1.1

Let $\mathcal{V}$ be a variety of groups, then $\mathcal{V}M(-)$ is a covariant functor from the category of all groups, $\mathcal{G}roups$, to the category of all abelian groups, $\mathcal{A}b$.

Proof. See [8] page 107.

Now with regards to the above theorem, we are going to concentrate on the relation between the functors, $M^{(c)}(-)$, $c \geq 1$, and the free product as follows.

In 1952, C. Miller [9] proved that $M(G) \cong H(G)$, where $H(G)$ is the group of all commutator relations of G, taken modulo universal commutator relations. He also showed that

Theorem 1.2 (C. Miller [9])

Let G_1 and G_2 be two arbitrary groups, then $H(G_1 * G_2) \cong H(G_1) \oplus H(G_2)$, where $G_1 * G_2$ is the free product of G_1 and G_2.

By the above theorem we can conclude the following corollary.

Corollary 1.3

The Schur multiplier functor, $M(-) : \mathcal{G}roups \longrightarrow \mathcal{A}b$, is coproduct-preserving. (Note that coproduct in $\mathcal{G}roups$ is free product and in $\mathcal{A}b$ is direct sum.)

In view of homology and cohomology of groups, we have the following theorem.

Theorem 1.4

Let A be a G-module, then $H^n(-, A)$, $H_n(-, A)$ are coproduct-preserving functors from $\mathcal{G}roups$ to $\mathcal{A}b$, for $n \geq 2$, i.e

$$H^n(G_1 * G_2, A) \cong H^n(G_1, A) \oplus H^n(G_2, A) \qquad for\ all\ n \geq 2\ ,$$

$$H_n(G_1 * G_2, A) \cong H_n(G_1, A) \oplus H_n(G_2, A) \qquad for\ all\ n \geq 2\ .$$

Proof. See [5, page 220].

Note that the above theorem does also confirm that the functor

$$M(-) = H_2(-, \mathbf{Z}) = H^2(-, \mathbf{C}^*)\ ,$$

is coproduct-preserving.

Now, with regards to the above theorems, it seems natural to ask whether the c-nilpotent multiplier functors $M^{(c)}(-)$, $c \geq 2$, are coproduct-preserving or not. To answer the question, first we state an important theorem of J. Burns and G. Ellis [2, Proposition 2.13 & Erratum at http://hamilton.ucg.ie/] which is proved by a homological method.

Theorem 1.5 (J. Burns and G. Ellis [2])

Let G and H be two arbitrary groups, then there is an isomorphism

$$M^{(2)}(G*H) \cong M^{(2)}(G) \oplus M^{(2)}(H) \oplus M(G) \otimes H^{ab} \oplus M(H) \otimes G^{ab} \oplus Tor(G^{ab}, H^{ab})\ ,$$

where $G^{ab} = G/G'$, $H^{ab} = H/H'$ and $Tor = Tor_1^{\mathbf{Z}}$.

Now, we are ready to show that the second nilpotent multiplier functor $M^{(2)}(-)$, *is not coproduct-preserving*, in general.

Example 1.6

Let $D_\infty =< a, b|a^2 = b^2 = 1 >\cong \mathbf{Z}_2 * \mathbf{Z}_2$ be the infinite dihedral group. Then

$$M^{(2)}(D_\infty) \ncong M^{(2)}(\mathbf{Z}_2) \oplus M^{(2)}(\mathbf{Z}_2) \quad .$$

Proof. By Theorem 1.5 we have

$$M^{(2)}(D_\infty) = M^{(2)}(\mathbf{Z}_2 * \mathbf{Z}_2)$$

$$\cong M^{(2)}(\mathbf{Z}_2) \oplus M^{(2)}(\mathbf{Z}_2) \oplus \mathbf{Z}_2 \otimes M(\mathbf{Z}_2) \oplus M(\mathbf{Z}_2) \otimes \mathbf{Z}_2 \oplus Tor(\mathbf{Z}_2, \mathbf{Z}_2) \ .$$

Clearly $M^{(2)}(\mathbf{Z}_2) = 0 = M(\mathbf{Z}_2)$. Also it is well-known that $Tor(\mathbf{Z}_2, \mathbf{Z}_2) \cong \mathbf{Z}_2 \otimes \mathbf{Z}_2 \cong \mathbf{Z}_2$ (see [11]). Therefore

$$M^{(2)}(\mathbf{Z}_2 * \mathbf{Z}_2) \cong \mathbf{Z}_2 \ ,$$

but

$$M^{(2)}(\mathbf{Z}_2) \oplus M^{(2)}(\mathbf{Z}_2) \cong 1 \quad .$$

Hence the result holds. □

In spite of the above example, using Theorem 1.5, we can show that the second nilpotent multiplier functor, $M^{(2)}(-)$, preserves the coproduct of a finite family of cyclic groups of mutually coprime order.

Corollary 1.7

Let $\{C_i|1 \le i \le n\}$ be a family of cyclic groups of mutually coprime order. Then

$$M^{(2)}(\prod_{i=1}^{n}{}^{*}C_i) \cong \oplus \sum_{i=1}^{n} M^{(2)}(C_i) \ ,$$

where $\prod_{i=1}^{n}{}^{*}C_i$ is the free product of C_i's, $1 \le i \le n$.

Proof. We proceed by induction on n. If $n = 2$, then by Theorem 1.5 and

using the fact that the Baer-invariant of any cyclic group is trivial, we have

$$M^{(2)}(C_1 * C_2) \cong Tor(C_1, C_2)\ .$$

Since C_1 and C_2 are finite abelian groups with coprime order, $Tor(C_1, C_2) \cong C_1 \otimes C_2 = 1$ (see [11]).

If $n = 3$, then similarly we have

$$M^{(2)}(C_1 * C_2 * C_3) \cong M^{(2)}(C_1 * C_2) \oplus M^{(2)}(C_3) \oplus M^{(1)}(C_1 * C_2) \otimes C_3$$

$$\oplus (C_1 * C_2)^{ab} \otimes M^{(1)}(C_3) \oplus Tor((C_1 * C_2)^{ab}, C_3)$$

$$\cong Tor(C_1 \oplus C_2, C_3) \cong (C_1 \oplus C_2) \otimes C_3 \cong (C_1 \otimes C_3) \oplus (C_2 \otimes C_3) = 1\ .$$

Note that $M^{(2)}(C_1 * C_2) = M^{(2)}(C_3) = M^{(1)}(C_1 * C_2) = 1$ By a similar procedure we can complete the induction. $\square$

2. The Main Result

In this section, we are going to generalize the above corollary to the variety of nilpotent groups of class at most c, $\mathcal{N}_c$, for all $c \geq 2$.

Notation 2.1

Let $C_i =< x_i | x_i^{r_i} > \cong \mathbf{Z}_{r_i}$ be cyclic group of order r_i , $1 \leq i \leq t$ such that $(r_i, r_j) = 1$ for all $i \neq j$. Put $C = \prod_{i=1}^{t} {}^* C_i$, the free product of C_i's, $1 \leq i \leq t$, $F = \prod_{i=1}^{t} {}^* F_i$, where F_i is the free group on $\{x_i\}$, $1 \leq i \leq t$, and $S =< x_i^{r_i} | 1 \leq i \leq t >^F$, the normal closure of $\{x_i^{r_i} | 1 \leq i \leq t\}$ in F. Note that F is free on $\{x_1, \dots x_t\}$. It is easy to see that the following sequence is exact.

$$1 \longrightarrow S \xrightarrow{\subseteq} F \xrightarrow{nat} C \longrightarrow 1\ .$$

Define by induction $\rho_1(S) = S$, $\rho_{n+1}(S) = [\rho_n(S), F]$. Now by Theorems 1.2 and 1.5, we have the following corollary.

Corollary 2.2

By the above notation and assumption, we have

(i) $S \cap \gamma_2(F) = \rho_2(S)$.

(ii) $S \cap \gamma_3(F) = \rho_3(S)$ and hence $\rho_2(S) \cap \gamma_3(F) = \rho_3(S)$.

Proof. (i) By Corollary 1.3 $M(C) = M(\prod_{i=1}^{t}{}^{*}C_i) \cong \oplus \sum_{i=1}^{t} M(C_i) = 1$. On the other hand, $M(C) \cong S \cap \gamma_2(F)/[S,F]$. Thus $S \cap \gamma_2(F)/[S,F] = 1$ and so $S \cap \gamma_2(F) = [S,F] = \rho_2(S)$.

(ii) By Corollary 1.7 $M^{(2)}(C) = M^{(2)}(\prod_{i=1}^{t}{}^{*}C_i) \cong \oplus \sum_{i=1}^{t} M^{(2)}(C_i) = 1$. Also by definition $M^{(2)}(C) \cong S \cap \gamma_3(F)/[S,_2 F]$, so $\cap\gamma_3(F) = [S,_2 F] = \rho_3(S)$. Moreover $\rho_3(S) \subseteq \rho_2(S) \cap \gamma_3(F) \subseteq S \cap \gamma_3(F) = \rho_3(S)$ and hence $\rho_2(S) \cap \gamma_3(F) = \rho_3(S)$. $\square$

Now we consider the following two technical lemmas.

Lemma 2.3

By the Notation 2.1 $\rho_n(S) \cap \gamma_{n+1}(F) = \rho_{n+1}(S)$, for all $n \geq 1$.

Proof. We proceed by induction on n. The assertion holds for $n = 1, 2$, by Corollary 2.2.

Now in order to avoid a lot of commutator manipulations, we prove the result for $n = 3$ in the special case $t = 2$. Put $x = x_1$, $y = x_2$, $r = r_1$, $s = r_2$. So F is free on $\{x, y\}$ and $S =< x^r, y^s >^F$.

Let g be a generator of $\rho_3(S)$, then

$$g = [(x^r)^{a_1}, y^{a_2}, x^{a_3}] \ or \ [(x^r)^{a_1}, y^{a_2}, y^{a_3}] \ or \ [(y^s)^{a_1}, x^{a_2}, y^{a_3}] \ or \ [(y^s), x^{a_2}, x^{a_3}] \ ,$$

where $a_i \in \mathbf{Z}$. Clearly modulo $\rho_4(S)$ we have

$$g \equiv [x^r, y, x]^{\alpha} \ or \ [x^r, y, y]^{\alpha} \ or \ [y^s, x, y]^{\alpha} \ or \ [y^s, x, x]^{\alpha} \ , \ where \ \alpha \in \mathbf{Z} \ .$$

Now, let $z \in \rho_3(S) \cap \gamma_4(F)$, then $z \in \rho_3(S)$. By the above fact and using a collecting process similar to basic commutators (see [3]) we can obtain the following congruence modulo $\rho_4(S)$

$$z \equiv [y^s, x, y]^{\alpha_1}[y, x^r, y]^{\beta_1}[y^s, x, x]^{\alpha_2}[y, x^r, x]^{\beta_2}$$

$$\equiv [y, x, y]^{s\alpha_1 + r\beta_1}[y, x, x]^{s\alpha_2 + r\beta_2} \ \ (mod \ \gamma_4(F)), \ where \ \alpha_i, \beta_i \in \mathbf{Z} \ .$$

Note that we consider the order on $\{x, y\}$ as $x < y$.

Since $z \in \rho_3(S) \cap \gamma_4(F)$ and $\rho_4(S) \subseteq \gamma_4(F)$, we have

$$[y, x, y]^{s\alpha_1 + r\beta_1}[y, x, x]^{s\alpha_2 + r\beta_2} \in \gamma_4(F) \ .$$

It is a well-known fact, by P. Hall [3, 4], that $\gamma_3(F)/\gamma_4(F)$ is the free abelian group on $\{[y, x, y], [y, x, x]\}$. Therefore we conclude that $s\alpha_i + r\beta_i = 0$, for $i = 1, 2$.

By a routine commutator calculation we have

$$[y^s, x, y]^{\alpha_1}[y, x^r, y]^{\beta_1} \equiv [[y^s, x]^{\alpha_1}[y, x^r]^{\beta_1}, y] \quad (mod\ \rho_4(S))$$

$$[y^s, x, x]^{\alpha_2}[y, x^r, x]^{\beta_2} \equiv [[y^s, x]^{\alpha_2}[y, x^r]^{\beta_2}, x] \quad (mod\ \rho_4(S)). \quad (*)$$

Also

$$[y, x]^{s\alpha_i + r\beta_i} \equiv [y^s, x]^{\alpha_i}[y, x^r]^{\beta_i} \in \rho_2(S) \ , \ for\ i = 1, 2 \ \ (mod \gamma_3(F)).$$

since $s\alpha_i + r\beta_i = 0$, $i = 1, 2$, we have

$$[y^s, x]^{\alpha_i}[y, x^r]^{\beta_i} \in \rho_2(S) \cap \gamma_3(F) \ , \ for\ i = 1, 2 \ .$$

By corollary 2.2 (ii) $\rho_2(S) \cap \gamma_3(F) = \rho_3(S)$, thus

$$[y^s, x]^{\alpha_i}[y, x^r]^{\beta_i} \in \rho_3(S) \ , \ for\ i = 1, 2 \ .$$

Therefore by $(*)$

$$[y^s, x, y]^{\alpha_1}[y, x^r, y]^{\beta_1} \quad , \quad [y^s, x, x]^{\alpha_2}[y, x^r, x]^{\beta_2} \in \rho_4(S)).$$

Hence $z \in \rho_4(S)$, and then $\rho_3(S) \cap \gamma_4(F) = \rho_4(S)$.

Note that by a similar method we can obtain the result for n, using induction hypothesis. $\square$

Lemma 2.4

By the above notation and assumption, $S \cap \gamma_n(F) = \rho_n(S)$, for all $n \geq 1$.

Proof. We proceed by induction on n. For $n = 1, 2$ Corollary 2.2 gives the result. Now, suppose $S \cap \gamma_n(F) = \rho_(S)$ for a natural number n. We show that $S \cap \gamma_{n+1}(F) = \rho_{n+1}(S)$.

Clearly $\rho_{n+1}(S) \subseteq S \cap \gamma_{n+1}(F)$, also $S \cap \gamma_{n+1}(F) \subseteq S \cap \gamma_n(F) = \rho_n(S)$, by induction hypothesis. Therefore by Lemma 2.3

$$\rho_{n+1}(S) \subseteq S \cap \gamma_{n+1}(F) \subseteq \rho_n(S) \cap \gamma_{n+1}(F) = \rho_{n+1}(S)\ .$$

Hence the result holds. □

Now, we are ready to show that the c-nilpotent multiplier functors, $\mathcal{N}_cM(-)$, preserve the coproduct of cyclic groups of mutually coprime order, for all $c \geq 1$.

Theorem 2.5

By the above notation and assumption,

$$M^{(c)}(\prod_{i=1}^{t}{}^{*}C_i) \cong \oplus \sum_{i=1}^{t} M^{(c)}(C_i) = 1\ ,\ for\ all\ c \geq 1\ .$$

Proof. By Lemma 2.4 and the definition of c-nilpotent multiplier, we have

$$M^{(c)}(\prod_{i=1}^{t}{}^{*}C_i) = \frac{S \cap \gamma_{c+1}(F)}{[S,_F]} = \frac{S \cap \gamma_{c+1}(F)}{\rho_{c+1}(S)} = 1\ ,\ for\ all\ c \geq 1\ .$$

On the other hand, since C_i's are cyclic, $M^{(c)}(C_i) = 1$, so $\oplus \sum_{i=1}^{t} \mathcal{N}_cM(C_i) = 1$, for all $c \geq 1$. Hence the result holds. □

Remark

In [2] it can be found some relations between the c-nilpotent multiplier and the c-isoclinism theory of P. Hall and also the notion of c-capable groups. Moreover, one may find in [2, page 423] a topological and also a homological interpretation of the c-nilpotent multiplier. Thus our result, Theorem 2.5, can be expressed and used in the above mentioned areas.

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